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NOTES ON THE FINITE ELEMENT ANALYSIS OF THE AXISYMMETRIC ELASTIC SOLID

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Abstract—A simple transformation of displacements considerably eases the explicit derivation of the finite element stiffness matrix for the axisymmetric elastic solid without causing a decline in the rate of convergence. The worsening of the condition of the global stiffness matrix caused by this transformation can be cured by scaling. A balanced numerical integration scheme maintaining the full rate of convergence is the one that integrates each term of the work and energy expressions to the order $2p-2$, p being the degree of the complete polynomial in the shape functions.

1. INTRODUCTION

Since the peculiarities of the axisymmetric state of strains are confined to one-dimension discussion is concentrated on this case. Here the radial and circumferential strains ε_r and ε_θ at a distance r from the origin are given in terms of the radial displacement u by $\varepsilon_r = u_r = \mathrm{d}u/\mathrm{d}r$ and $\varepsilon_\theta = u/r$. Appropriately, the accuracy of the approximate solution is measured in the energy norm $\|\cdot\|_1$

$$\|u\|_1{}^2 = \int_0^1 (\varepsilon_r{}^2 + \varepsilon_\theta{}^2) r \,\mathrm{d}r. \tag{1}$$

The standard finite element technique[1–10] for the construction of the stiffness matrix out of the energy expression (1) consists of assuming u to be polynomial inside the element and forming the entires of the matrix by integration. The appearance of the rational coefficient $1/r$ in it greatly encumbers the explicit integration of the circumferential energy term, particularly in space where integration needs be carried out over triangles. Use of a numerical integration, which is discussed later on, alleviates the tedium of the algebraic integrations but the main purpose of this paper is to show that the transformation $v = u/r$, $\varepsilon_\theta = v$, $\varepsilon_r = (vr)_{,r}$, which removes $1/r$ from the energy, with a polynomial approximation for v does not result in a loss of accuracy.

2. DISCRETIZATION ACCURACY

To keep matters as simple and explicit as possible a detailed analysis is carried out here only on the first order elements. Extension of the results to higher order elements is a formality. So first u is assumed to be approximated linearly inside the element $r_1 \le r \le r_2$ by $\tilde{u}$ in the form

$$\tilde{u} = [u_1(r_2 - r_1) + u_2(r - r_1)]/h \tag{2}$$

where $h = r_2 - r_1$, and where u_1 and u_2 are the values of u at the nodal points at r_1 ad r_2. If $\hat{u}$ denotes the finite element solution then $\|u - \hat{u}\|_1 \le \|u - \tilde{u}\|_1$ and hence to bound the error in the finite element solution it is sufficient to estimate $\|u - \tilde{u}\|_1$ for any reasonable $\tilde{u}$. Following Synge[11], $\tilde{u}$ for that purpose is selected to be the finite element interpolate to $\dot{u}$ such that at the nodes $u = \tilde{u}$. Also since $\|u - \tilde{u}\|_1{}^2$ is equal to the sum of similar terms over all the individual elements only the typical element errors e_1 and e_2

$$e_1{}^2 = \int_{r_1}^{r_2} (u_r - \hat{u}_r)^2 r \, dr, \qquad e_2{}^2 = \int_{r_1}^{r_2} (u^2/r^2 - \hat{u}^2/r^2) r \, dr \tag{3}$$

need be evaluated.

To bound $e_1{}^2$, $\tilde{u}$ in equation (2) is differentiated yielding $\tilde{u}_r = (u_2 - u_1)/h$. When u_2 is expanded in a Taylor series with a remainder around r_1 this yields $\tilde{u}_r = u_1' + 0(hu'')$ where $(\ )'$ stands for differentiating with respect to r like $(\ )_r$. Expansion of the true solution u around r_1 produces $u_r = u_1' + 0[(r_2 - r)u'']$ and hence

$$\max|u_r - \tilde{u}_r| \le ch \max|u''| \tag{4}$$

where c is a generic constant independent of h and u and where the maximum is sought over $r_1 \le r \le r_2$. Introduction of equation (4) into $e_1{}^2$ in equation (3) results in

$$e_1{}^2 \le c^2 h^2 \max|u''|^2 (r_1 + r_2)h/2. \tag{5}$$

The estimation of $e_2{}^2$ is rather simple for $r_1 > 0$ since then

$$e_2{}^2 \le \frac{1}{r_1{}^2} \max|u - \tilde{u}|^2 (r_1 + r_2)h/2 \tag{6}$$

which with $\tilde{u} = \tilde{u}_1 + (r - r_1)u_1' + (r - r_1)0(hu'')$ and $u = u_1 + (r - r_1)u_1' + 0[(r - r_1)^2 u'']$ readily becomes

$$e_1{}^2 \le \frac{c^2}{r_1{}^2} h^4 \max|u''|^2 (r_1 + r_2)h/2. \tag{7}$$

The element nearest to the origin requires a special treatment. In it $\tilde{u} = ru_2/h$ and since $r = 0$ is not a singular point[12] u_1 must vanish there. Hence in this element $\tilde{u}/r = u_1' + 0(hu'')$, $u/r = u_1' + 0(ru'')$ and consequently $\max|u/r - \tilde{u}/r| \le ch \max|u''|$ leading to $e_2{}^2 \le c^2 h^2 \max|u''|^2 h^2/2$ in the first element. Summation of $e_1{}^2$ and $e_2{}^2$ over all the finite elements yields

$$\|u - \hat{u}\|_1 \le ch \max|u''| \tag{8}$$

where $\max|\quad|$ is now over $0 \le r \le 1$.

When u/r is replaced v

$$e_1{}^2 = \int_{r_1}^{r_2} [(vr)_r - (\tilde{v}r)_r]^2 r \, dr, \qquad e_2{}^2 = \int_{r_1}^{r_2} (v - \tilde{v})^2 r \, dr. \tag{9}$$

For a linear approximation for $\tilde{v}$ it was previously obtained for $e_2{}^2$ that

$$e_2{}^2 \le c^2 h^4 \max|v''|^2 (r_1 + r_2)h/2 \tag{10}$$

when v replaces u in equation (2), $(rv)_r = [v_1 h + + (2r - r_1)(v_2 - v_1)]/h$ which with Taylor's theorem becomes $(r\tilde{v})_r = v_1 + [2r - r_1)(v_1' + 0(hv'')]$. On the other hand $(rv)_r = (rv)_1' + (r - r_1)0[(rv)'']$ and consequently since $|r - r_1| \le h$ the yields that

$$\max|(rv) - (r\tilde{v})_r| \le ch \max|v', v''| \tag{11}$$

summation of the errors $e_1{}^2$ and $e_2{}^2$ over all the finite elements with $u = vr$ produces the error estimate

$$\|u - \hat{u}\|_1 \le ch \max|\varepsilon_\theta', \varepsilon_\theta''| \tag{12}$$

and the error in the energy norm is again $0(h)$ since ε_θ and its derivatives are bounded.

In the more general case when u or v are approximated by a polynomial of degree p the discretization error in the energy norm is $0(h^p)$. The proof to that is entirely analogous to that just given for $p = 1$, except that more terms need be taken in the Taylor expansions.

The preservation of the rate of convergence with $v = u/r$ in spite of the lack of a constant term (the rigid body mode) in the shape functions for v can be explained in different ways. For one, these shape functions which are $(1 - \xi)(r_1 + h\xi)$ and $\xi(r_1 + h\xi)$ where $\xi = (r - r_1)/h$, recover the complete linear polynomial away from the origin as $h \to 0$; and near it the exact solution itself lacks a constant due to symmetry. Also, it is immaterial which variables appear in the energy expression as long as it is polynomial-like in the sense that it can be expanded in a polynomial power series to a sufficiently high degree. Here, due to the regularity of the solution both u and u/r are polynomial-like and assuming a polynomial approximation of degree p for either u or u/r produces the same error $0(h^{2p})$.

3. CONDITION OF GLOBAL STIFFNESS MATRIX

In this axisymmetric case the element stiffness matrix k_e is positive definite and the spectral condition number $C_2(K)$ of K can be bounded[13, 14] in this case by

$$C_2(K) \leq p_{\max} \max_e(\lambda_n{}^{k_e})/\min_e(\lambda_1{}^{k_e}) \tag{13}$$

where $\lambda_1{}^k$ and $\lambda_n{}^k$ denote the extremal eigenvalues of k, $p_{\max}$ the maximum number of elements meeting at a nodal point and where $\max_e(\ )$ denotes the maximum value over all finite elements.

In second order problems $C_2(K) = 0(h^{-2})$. Using the transformation $v = u/r$ it increases to $C_2(K) = 0(h^{-3})$, but with the aid of equation (13) it can be shown that scaling, which makes all K_{ii} equal, reduces $C_2(K)$ back to the optimal $0(h^{-2})$.

4. NUMERICAL INTEGRATION

It has been shown in[15] that if the shape functions include a complete polynomial of degree p then the full rate of convergence is maintained if each term in the work, and potential and kinetic energies is integrated numerically to the degree $2(p-m)$, m being the highest order of differentiation in the energy expressions. Here $m = 1$ and hence each term need be integrated to the degree $2p - 2$. For instance, if the shape functions include a quadratic ($p = 2$) then a quadratic integration scheme is required and a 2 Gauss points rule, which is already cubic, is sufficient.

5. NUMERICAL EXAMPLE

The theoretical predictions of the revious sections is tested numerically on the vibrating elastic disc. As in statics also in dynamics[16], if the shape function, in second order problems, include a complete polynomial of degree p then the error in the eigenvalues is $0(h^{2p})$. Figure 1 shows the convergence of the fundamental eigenvalue λ in the vibrating disc vs the number of elements $N_e = h^{-1}$ along the radius and indeed the rates of convergence with a linear approximation to u (curve a) and a linear approximation to $v = u/r$ (curve c) are nearly 2. The better results obtained with the transformation $v = u/r$ can be explained by the fact that in this case no conditions need be imposed on v at $r = 0$ while without the transformation, u must be set equal to zero at the origin $r = 0$ to avoid the blow-up of the stiffness matrix, causing a loss of one degree of freedom. Curve (b) in Fig. 1 refers to numerical integration of the circumferential energy with one Gauss point which according to section 4 should suffice, as indeed it does, to maintain the full rate of convergence.

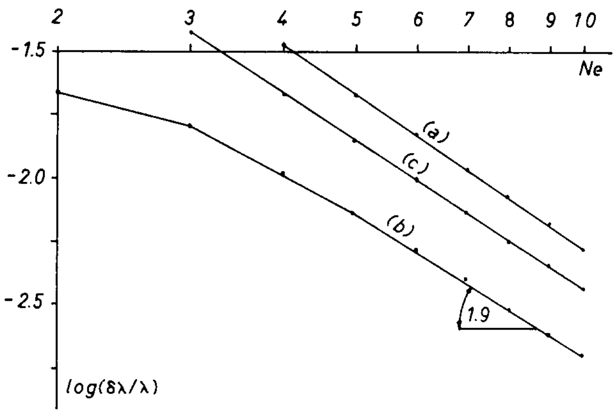

Fig. 1. Error $\delta\lambda$ in fundamental eigenvalue λ of an elastic disc held fix at its rim vs the number of elements *Ne* along the radius for; (a) a linear approximation of u; (b) a linear approximation of $v = u/r$; and (c) a linear approximation of u but with the circumferential energy integrated numerically with one Gauss point.

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Абстракт — Несложное преобразование перемещений значительно облегчает вывод в явной форме матрицы коэффициентов жесткости конечного элемента, для осесимметрического упругого тела, без вызвания снижения скорости сходимости. Ухудшение условия, рассматриваемой в целом, матрицы коэффициентов жесткости, по поводу этого преобразования, можно исправлять путем сведения к определенному масштабу. Уравновешенная схема численного интегрирования, содержащая полную скорость сходимости такого же вида, что интегрируется каждый член выражений работы и энергии до порядка $2p - 2$, где p является порядком полного полинома функций формы.